BAL BHARATI PUBLIC SCHOOL PITAMPURA

Session : 2020-21

Class 12

Mathematics

CHAPTER 5

Continuity & Differentiability (Part 6)

Reference Book:
NCERT Class 12(Part 1)
http://ncert.nic.in/textbook/textbook.htm?lemh1=5-6

Differentiation of function with respect to another function

Find the derivative of $\tan^{-1}(1+x^2)$ with respect to x^2+x+1.

Solution :

Let $f(x) = \tan^{-1}(1+x^2)$

$g(x) = x^2+x+1$

Let

$$f(x) = \tan^{-1}(1+x^2)$$

$$g(x) = x^2+x+1$$

$$\frac{df}{dg} = \frac{f'(x)}{g'(x)}$$

$$f'(x) = \frac{2x}{1+x^2}$$

$$g'(x) = 2x+1$$

$$\frac{df}{dg} = \frac{\frac{2x}{1+x^2}}{2x+1} = \frac{2x}{(2x+1)(1+x^2)}.$$

Differentiate $\sin(ax^2 + bx + c)$ with respect to $\cos(lx^2 + mx + n)$

Solution

Let $u = \sin(ax^2 + bx + c)$

$v = \cos(lx^2 + mx + n)$

$$\begin{aligned}
\text{Let} \quad u &= \sin(ax^2 + bx + c) \\
v &= \cos(lx^2 + mx + n) \\
\frac{du}{dv} &= \frac{u'(x)}{v'(x)} \\
u'(x) &= \cos(ax^2 + bx + c)(2ax + b) \\
v'(x) &= -\sin(lx^2 + mx + n)(2lx + m) \\
\frac{du}{dv} &= \frac{u'(x)}{v'(x)} = \frac{(2ax + b)\cos(ax^2 + bx + c)}{-(2lx + m)\sin(lx^2 + mx + n)} .
\end{aligned}$$

Assignment 7

$Differentiate \quad \log\sin x \ with\,respect\,to\ \sqrt{\cos x}$
$Differentiate \quad \tan^{-1}(\frac{1+2x}{1-2x})\ with\,respect\,to \sqrt{1+4x^2}$
$Differentiate \quad \tan^{-1}(\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}})\ with\,respect\,to\ \cos^{-1}x^2$
$Differentiate \quad x^{\sin^{-1}x}\ with\,respect\,to\ \sin^{-1}x$
$Differentiate \quad \sin^{-1}(\frac{2x}{1+x^2})\,with\,respect\,to\ \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)if\ 0<x<1$
$Differentiate \quad \tan^{-1}(\frac{1+ax}{1-ax})\ with\,respect\,to \sqrt{1+a^2x^2}$
$Differentiate \quad \sin^{-1}(2ax\sqrt{1-a^2x^2})\ with\,respect\,to \sqrt{1-a^2x^2}\ \ if -\frac{1}{\sqrt{2}}<ax<\frac{1}{\sqrt{2}}$
$Differentiate \quad \cos^{-1}(4x^3-3x)\ with\,respect\,to\ \tan^{-1}(\frac{\sqrt{1-x^2}}{x})\ if\frac{1}{2}<x<1$

Practice Questions

For a positive constant a find $\frac{dy}{dx}$, where

$$y = a^{t+\frac{1}{t}}, \text{ and } x = \left(t + \frac{1}{t}\right)^a$$

Solution Observe that both y and x are defined for all real $t \neq 0$. Clearly

$$\frac{dy}{dt} = \frac{d}{dt}\left(a^{t+\frac{1}{t}}\right) = a^{t+\frac{1}{t}} \frac{d}{dt}\left(t+\frac{1}{t}\right) \cdot \log a$$

$$= a^{t+\frac{1}{t}}\left(1-\frac{1}{t^2}\right)\log a$$

Similarly

$$\frac{dx}{dt} = a\left[t+\frac{1}{t}\right]^{a-1} \cdot \frac{d}{dt}\left(t+\frac{1}{t}\right)$$

$$= a\left[t+\frac{1}{t}\right]^{a-1} \cdot \left(1-\frac{1}{t^2}\right)$$

$\frac{dx}{dt} \neq 0$ only if $t \neq \pm 1$. Thus for $t \neq \pm 1$,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a^{t+\frac{1}{t}}\left(1-\frac{1}{t^2}\right)\log a}{a\left[t+\frac{1}{t}\right]^{a-1} \cdot \left(1-\frac{1}{t^2}\right)}$$

$$= \frac{a^{t+\frac{1}{t}}\log a}{a\left(t+\frac{1}{t}\right)^{a-1}}$$

Differentiate $\sin^2 x$ w.r.t. $e^{\cos x}$.

Solution Let $u(x) = \sin^2 x$ and $v(x) = e^{\cos x}$. We want to find $\frac{du}{dv} = \frac{du/dx}{dv/dx}$. Clearly

$$\frac{du}{dx} = 2 \sin x \cos x \text{ and } \frac{dv}{dx} = e^{\cos x}(-\sin x) = -(\sin x)\, e^{\cos x}$$

Thus

$$\frac{du}{dv} = \frac{2 \sin x \cos x}{-\sin x\, e^{\cos x}} = -\frac{2 \cos x}{e^{\cos x}}$$

TASK

- Do examples and Misc Exercise from NCERT

Miscellaneous Exercise on Chapter 5

Differentiate w.r.t. x the function in Exercises 1 to 11.

1. $(3x^2 - 9x + 5)^9$
2. $\sin^3 x + \cos^6 x$
3. $(5x)^{3\cos 2x}$
4. $\sin^{-1}(x\sqrt{x}),\ 0 \le x \le 1$
5. $\dfrac{\cos^{-1}\dfrac{x}{2}}{\sqrt{2x+7}},\ -2 < x < 2$
6. $\cot^{-1}\left[\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right],\ 0 < x < \dfrac{\pi}{2}$
7. $(\log x)^{\log x},\ x > 1$
8. $\cos(a\cos x + b\sin x)$, for some constant a and b.
9. $(\sin x - \cos x)^{(\sin x - \cos x)},\ \dfrac{\pi}{4} < x < \dfrac{3\pi}{4}$
10. $x^x + x^a + a^x + a^a$, for some fixed $a > 0$ and $x > 0$
11. $x^{x^2-3} + (x-3)^{x^2}$, for $x > 3$
12. Find $\dfrac{dy}{dx}$, if $y = 12(1 - \cos t),\ x = 10(t - \sin t),\ -\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$
13. Find $\dfrac{dy}{dx}$, if $y = \sin^{-1} x + \sin^{-1}\sqrt{1-x^2},\ 0 < x < 1$
14. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, for $-1 < x < 1$, prove that

$$\frac{dy}{dx} = -\frac{1}{(1+x)^2}$$

15. If $(x-a)^2 + (y-b)^2 = c^2$, for some $c > 0$, prove that

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

is a constant independent of a and b.

16. If $\cos y = x \cos (a + y)$, with $\cos a \neq \pm 1$, prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$.

17. If $x = a(\cos t + t \sin t)$ and $y = a(\sin t - t \cos t)$, find $\frac{d^2y}{dx^2}$.

18. If $f(x) = |x|^3$, show that $f''(x)$ exists for all real x and find it.

19. Using mathematical induction prove that $\frac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers n.

20. Using the fact that $\sin(A+B) = \sin A \cos B + \cos A \sin B$ and the differentiation, obtain the sum formula for cosines.

21. Does there exist a function which is continuous everywhere but not differentiable at exactly two points? Justify your answer.

22. If $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$, prove that $\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

23. If $y = e^{a\cos^{-1}x}$, $-1 \leq x \leq 1$, show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$.