ALGEBRA

(Factorisation)

LEARNING OBJECTIVES

- Students will be able recall various laws of indices
- Students will be able recall various algebraic identities and factorise algebraic expressions using them.
- Students will be able recall various methods of solving quadratic equations.

TOPICS

- Laws Of Indices
- Algebraic Identities used for factorisation
- Quadratic Equations
  - ✓ Factorisation using Middle Term Splitting
  - ✓ Factorisation using Completing Whole Square
  - ✓ Factorisation using Factor Theorem
  - ✓ Discriminant Method of Solving Equations
  - ✓ Introduction of i(iota)

CONTENT

Laws of Indices

Law	Example
$x^1 = x$	$6^1 = 6$
$x^0 = 1$	$7^0 = 1$
$x^{-1} = 1/x$	$4^{-1} = 1/4$
$x^m x^n = x^{m+n}$	$x^2 x^3 = x^{2+3} = x^5$
$x^m/x^n = x^{m-n}$	$x^6/x^2 = x^{6-2} = x^4$
$(x^m)^n = x^{mn}$	$(x^2)^3 = x^{2\times 3} = x^6$
$(xy)^n = x^n y^n$	$(xy)^3 = x^3 y^3$
$(x/y)^n = x^n/y^n$	$(x/y)^2 = x^2 / y^2$
$x^{-n} = 1/x^n$	$x^{-3} = 1/x^3$
And the law about Fractional Exponents:	
$x^{\frac{m}{n}} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$	$x^{\frac{2}{3}} = \sqrt[3]{x^2} = (\sqrt[3]{x})^2$

Think????

Find the value of the following

- $\sqrt{-1} \times \sqrt{-1}$
- $\sqrt{1} \times \sqrt{-1}$
- $\sqrt{1} \times \sqrt{1}$

Oh, One More Thing ... What if x = 0?

Positive Exponent (n>0)	$0^n = 0$
Negative Exponent (n<0)	0^{-n} is **undefined** (because dividing by 0 is undefined)
Exponent = 0	0^0 ... *ummm* ... see below!

The Strange Case of 0^0

There are different arguments for the correct value of 0^0

0^0 could be 1, or possibly 0, so some people say it is really "indeterminate":

$x^0 = 1$, so ...	$0^0 = 1$
$0^n = 0$, so ...	$0^0 = 0$
When in doubt ...	$0^0 =$ *"indeterminate"*

Think??

Is undefined/not defined same as indeterminate?

The process of expressing a given algebraic expression as a product of algebraic expressions, each of degree less than that of given algebraic expression, is called **factorization.**

There are large numbers of topics in class XI where you need to apply various methods of factorizing polynomials like splitting the middle term, Using Factor Theorem, zeroes of a polynomial and Quadratic Formula.

Some of such topics in class XI are Relations and functions, Complex Numbers, Permutations and Combinations, Binomial Theorem, Sequence and Series, Straight Lines, Conic Sections, Limits and Derivatives etc. So, here we go!!

Algebraic Identities

Identity

An **identity** is an equality that holds true regardless of the values chosen for its variables. They are used in simplifying or rearranging algebra expressions. For example, the identity $(a+b)^2 = a^2 + 2ab + b^2$ is true for all the values chosen for a and b.

Some other identities are:

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$a^2 - b^2 = (a+b)(a-b)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)\left(a^2 + b^2 + c^2 - ab - bc - ca\right)$$

Factorization using identities

: Factorize $x^8 - y^8$

Solution:

$$x^8 - y^8$$
$$= \left(x^4\right)^2 - \left(y^4\right)^2$$
$$= \left(x^4 + y^4\right)\left(x^4 - y^4\right)$$
$$= \left(x^4 + y^4\right)\left[\left(x^2\right)^2 - \left(y^2\right)^2\right]$$
$$= \left(x^4 + y^4\right)\left(x^2 + y^2\right)\left(x^2 - y^2\right)$$
$$= \left(x^4 + y^4\right)\left(x^2 + y^2\right)(x+y)(x-y)$$

Quadratic Equation

Factorisation using splitting the middle term

Let x^2+bx+c be a quadratic polynomial and $(x+p)$ and $(x+q)$ are other tw
polynomials such that

$$x^2+bx+c=(x+p)(x+q)$$
$$=x^2+(p+q)x+pq$$

Comparing the coefficients of like terms, we get $p+q=b$ and $pq=c$

So, in order to factorize quadratic polynomial x^2+bx+c, we have to find two terr
p and q such that $p+q=b$ and $pq=c$.

Let us understand this process by considering an example.

Factorize: $x^2-7x+12$

Solution:

We note 12 = (-3) x (-4) and (-3) + (-4) = -7. Therefore,

$$x^2-7x+12=x^2-3x-4x+12=x(x-3)-4(x-3)=(x-3)(x-4)$$

Factorisation using completing the square

Factorize $x^2-7x-12$

Answer: Consider $x^2-7x-12$

adding $\left(\frac{1}{2}\text{coeff. of } x\right)^2$ i.e. $\left[\frac{1}{2}\times(-7)\right]^2=\frac{49}{4}$ to both sides, we get

$$x^2-7x+\frac{49}{4}-\frac{49}{4}-12$$
$$=\left(x-\frac{7}{2}\right)^2-\frac{97}{4}$$
$$=\left(x-\frac{7}{2}\right)^2-\left(\frac{\sqrt{97}}{2}\right)^2$$
$$=\left(x-\frac{7}{2}+\frac{\sqrt{97}}{2}\right)\left(x-\frac{7}{2}-\frac{\sqrt{97}}{2}\right)$$

Example:

$$x^2 + y^2 + 6x - 4y - 12 = 0$$

Step 1 - Commute and associate the x and y terms; additive inverse the -12:

$$(x^2 + 6x) + (y^2 - 4y) = 12$$

Step 2 - Complete the squares, (what you do to one side be sure to do to the other side):

$$(x^2 + 6x + 9) + (y^2 - 4y + 4) = 12 + 9 + 4$$

Step 3 - Factor:

$$(x + 3)^2 + (y - 2)^2 = 25 = 5^2$$

Factorisation using Factor theorem

Factor Theorem: If $p(x)$ is a polynomial of degree $n \geq 1$ and a is any real number, then

(i) $x-a$ is a factor of $p(x)$, if $p(a)=0$, and

(ii) $p(a)=0$, if $x-a$ is a factor of $p(x)$

Factor theorem can be used to factorize some polynomials. For this purpose, we try to find an integral zero of the polynomial. As soon as we get a zero of a polynomial, we get one of the factors of the polynomial. We repeat this process until we get a polynomial which can be further factorized easily.

Using factor theorem, factorize the polynomial $x^3 - 6x^2 + 11x - 6$.

Let $f(x) = x^3 - 6x^2 + 11x - 6$

The constant term in f(x) is equal to – 6 and factors of – 6 are $\pm 1, \pm 2, \pm 3, \pm 6$.

Putting x = 1 in f(x), we have

$f(1) = 1^3 - 6 \times 1^2 + 11 \times 1 - 6$

$= 1 - 6 + 11 - 6 = 0$

$\therefore$ (x– 1) is a factor of f(x)

Similarly, x – 2 and x – 3 are factors of f(x).

Since f(x) is a polynomial of degree 3.

So, it can not have more than three linear factors.

Let f(x) = k (x–1) (x– 2) (x – 3). Then,

$x^3 - 6x^2 + 11x - 6 = k(x-1)(x-2)(x-3)$

Putting x = 0 on both sides, we get

$-6 = k(0-1)(0-2)(0-3)$

$\Rightarrow -6 = -6k \Rightarrow k = 1$

Putting k = 1 in f(x) = k (x– 1) (x– 2) (x–3), we get

f(x) = (x–1) (x– 2) (x – 3)

Hence, $x^3 - 6x^2 + 11x - 6 = (x-1)(x-2)(x-3)$

Note :When only one or two roots can be found using factor theorem then we do the following procedure

Let $f(x) = 2x^4 + x^3 - 14x^2 - 19x - 6$ be the given polynomial.

The factors of the constant term -6 are $\pm1, \pm2, \pm3$ and ±6, we have,

$f(-1) = 2(-1)^4 + (-1)^3 - 14(-1)^2 - 19(-1) - 6$

$= 2 - 1 - 14 + 19 - 6 = 21 - 21 = 0$

and,

$f(-2) = 2(-2)^4 + (-2)^3 - 14(-2)^2 - 19(-2) - 6$

$= 32 - 8 - 56 + 38 - 6 = 0$

So, $x + 1$ and $x + 2$ are factors of $f(x)$.

$\Rightarrow$ $(x + 1)(x + 2)$ is also a factor of $f(x)$

$\Rightarrow$ $x^2 + 3x + 2$ is a factor of $f(x)$

Now, we divide

$f(x) = 2x^4 + x^3 - 14x^2 - 19x - 6$ by

$x^2 + 3x + 2$ to get the other factors.

Discriminant Method of solving quadratic equations

Many quadratic equations cannot be solved by factoring. This is generally true when the roots, or answers, are not rational numbers. Another method of solving quadratic equations $ax^2 + bx + c = 0$ involves the use of the following formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

When using the quadratic formula, you should be aware of three possibilities.

These three possibilities are distinguished by a part of the formula called the Discriminant.

The *Discriminant* is the value under the square root sign, $b^2 - 4ac$. A quadratic equation with real numbers as coefficients can have the following:

1. Two different real roots if the Discriminant $b^2 - 4ac > 0$.
2. One real root if the Discriminant $b^2 - 4ac = 0$.
3. No real root if the Discriminant $b^2 - 4ac < 0$.

Example :Solve $5x^2 + 6x + 1 = 0$

Coefficients are: $a = 5,\ b = 6,\ c = 1$

Quadratic Formula: $x = \dfrac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$

Put in a, b and c: $x = \dfrac{-6 \pm \sqrt{(6^2 - 4\times5\times1)}}{2\times5}$

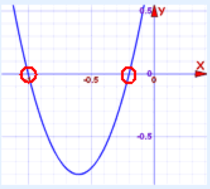
Solve: $x = \dfrac{-6 \pm \sqrt{(36 - 20)}}{10}$

$x = \dfrac{-6 \pm \sqrt{(16)}}{10}$

$x = \dfrac{-6 \pm 4}{10}$

$x = -0.2$ **or** -1

Answer: $x = -0.2$ **or** $x = -1$



And we see them on this graph.

Notice in the above method we were not able to solve when the square root had a negative sign inside it. Thus a lot of polynomials remained unsolved. In order to solve polynomials of such kind, we introduced a new concept.

Introduction to i(iota)

The "unit" imaginary number (like 1 for Real Numbers) is i, which is the square root of −1

$$i = \sqrt{-1}$$

Because when we square i we get −1

$$i^2 = -1$$

$$\sqrt{-a} = \sqrt{-1 \cdot a} = \sqrt{-1} \cdot \sqrt{a} = i\sqrt{a}$$

And hence we introduce complex numbers (which are bigger than real numbers)

A Complex Number is a combination of a Real Number and an Imaginary Number: