

# DETERMINANTS

## PART 2

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# PROPERTIES OF DETERMINANTS

**Property 1** The value of the determinant remains unchanged if its rows and columns are interchanged.

**Verification** Let  $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Expanding along first row, we get

$$\begin{aligned} \Delta &= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\ &= a_1 (b_2 c_3 - b_3 c_2) - a_2 (b_1 c_3 - b_3 c_1) + a_3 (b_1 c_2 - b_2 c_1) \end{aligned}$$

By interchanging the rows and columns of  $\Delta$ , we get the determinant

$$\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Expanding  $\Delta_1$  along first column, we get

$$\Delta_1 = a_1 (b_2 c_3 - c_2 b_3) - a_2 (b_1 c_3 - b_3 c_1) + a_3 (b_1 c_2 - b_2 c_1)$$

Hence  $\Delta = \Delta_1$

*Remark* It follows from above property that if  $A$  is a square matrix, then  $\det(A) = \det(A')$ , where  $A'$  = transpose of  $A$ .

**Note** If  $R_i = i$ th row and  $C_j = j$ th column, then for interchange of row and columns, we will symbolically write  $C \leftrightarrow R$ .

**Property 2** If any two rows (or columns) of a determinant are interchanged, then sign of determinant changes.

**Verification** Let  $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Expanding along first row, we get

$$\Delta = a_1 (b_2 c_3 - b_3 c_2) - a_2 (b_1 c_3 - b_3 c_1) + a_3 (b_1 c_2 - b_2 c_1)$$

Interchanging first and third rows, the new determinant obtained is given by

$$\Delta_1 = \begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

Expanding along third row, we get

$$\begin{aligned} \Delta_1 &= a_1 (c_2 b_3 - b_2 c_3) - a_2 (c_1 b_3 - c_3 b_1) + a_3 (b_2 c_1 - b_1 c_2) \\ &= - [a_1 (b_2 c_3 - b_3 c_2) - a_2 (b_1 c_3 - b_3 c_1) + a_3 (b_1 c_2 - b_2 c_1)] \end{aligned}$$

Clearly  $\Delta_1 = -\Delta$

Similarly, we can verify the result by interchanging any two columns.

**Note** We can denote the interchange of rows by  $R_i \leftrightarrow R_j$  and interchange of columns by  $C_i \leftrightarrow C_j$ .

**Property 3** If any two rows (or columns) of a determinant are identical (all corresponding elements are same), then value of determinant is zero.

**Proof** If we interchange the identical rows (or columns) of the determinant  $\Delta$ , then  $\Delta$  does not change. However, by Property 2, it follows that  $\Delta$  has changed its sign

Therefore  $\Delta = -\Delta$

or  $\Delta = 0$

**Property 4** If each element of a row (or a column) of a determinant is multiplied by a constant  $k$ , then its value gets multiplied by  $k$ .

**Verification** Let  $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

and  $\Delta_1$  be the determinant obtained by multiplying the elements of the first row by  $k$ .  
Then

$$\Delta_1 = \begin{vmatrix} k a_1 & k b_1 & k c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Expanding along first row, we get

$$\begin{aligned} \Delta_1 &= k a_1 (b_2 c_3 - b_3 c_2) - k b_1 (a_2 c_3 - c_2 a_3) + k c_1 (a_2 b_3 - b_2 a_3) \\ &= k [a_1 (b_2 c_3 - b_3 c_2) - b_1 (a_2 c_3 - c_2 a_3) + c_1 (a_2 b_3 - b_2 a_3)] \\ &= k \Delta \end{aligned}$$

NOTE :

*Remarks*

- (i) By this property, we can take out any common factor from any one row or any one column of a given determinant.
- (ii) If corresponding elements of any two rows (or columns) of a determinant are proportional (in the same ratio), then its value is zero. For example

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ k a_1 & k a_2 & k a_3 \end{vmatrix} = 0 \text{ (rows } R_1 \text{ and } R_2 \text{ are proportional)}$$

**Property 5** If some or all elements of a row or column of a determinant are expressed as sum of two (or more) terms, then the determinant can be expressed as sum of two (or more) determinants.

For example, 
$$\begin{vmatrix} a_1 + \lambda_1 & a_2 + \lambda_2 & a_3 + \lambda_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

**Verification L.H.S.** = 
$$\begin{vmatrix} a_1 + \lambda_1 & a_2 + \lambda_2 & a_3 + \lambda_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Expanding the determinants along the first row, we get

$$\begin{aligned} \Delta &= (a_1 + \lambda_1) (b_2 c_3 - c_2 b_3) - (a_2 + \lambda_2) (b_1 c_3 - b_3 c_1) \\ &\quad + (a_3 + \lambda_3) (b_1 c_2 - b_2 c_1) \\ &= a_1 (b_2 c_3 - c_2 b_3) - a_2 (b_1 c_3 - b_3 c_1) + a_3 (b_1 c_2 - b_2 c_1) \\ &\quad + \lambda_1 (b_2 c_3 - c_2 b_3) - \lambda_2 (b_1 c_3 - b_3 c_1) + \lambda_3 (b_1 c_2 - b_2 c_1) \end{aligned}$$

(by rearranging terms)

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \text{R.H.S.}$$

**Property 6** If, to each element of any row or column of a determinant, the equimultiples of corresponding elements of other row (or column) are added, then value of determinant remains the same, i.e., the value of determinant remain same if we apply the operation  $R_i \rightarrow R_i + kR_j$  or  $C_i \rightarrow C_i + kC_j$ .

**Verification**

Let 
$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \text{ and } \Delta_1 = \begin{vmatrix} a_1 + kc_1 & a_2 + kc_2 & a_3 + kc_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix},$$

where  $\Delta_1$  is obtained by the operation  $R_1 \rightarrow R_1 + kR_3$ .

Here, we have multiplied the elements of the third row ( $R_3$ ) by a constant  $k$  and added them to the corresponding elements of the first row ( $R_1$ ).

Symbolically, we write this operation as  $R_1 \rightarrow R_1 + kR_3$ .

Now, again

$$\Delta_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} kc_1 & kc_2 & kc_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad (\text{Using Property 5})$$

$$= \Delta + 0 \quad (\text{since } R_1 \text{ and } R_3 \text{ are proportional})$$

Hence  $\Delta = \Delta_1$

**Remarks**

- (i) If  $\Delta_1$  is the determinant obtained by applying  $R_i \rightarrow kR_i$  or  $C_i \rightarrow kC_i$  to the determinant  $\Delta$ , then  $\Delta_1 = k\Delta$ .
- (ii) If more than one operation like  $R_i \rightarrow R_i + kR_j$  is done in one step, care should be taken to see that a row that is affected in one operation should not be used in another operation. A similar remark applies to column operations.

# TASK

- Watch the video:  
<https://youtu.be/jGReqvTUHwI>
- Watch the video to learn how to solve properties of determinants questions:  
<https://youtu.be/K AB7thbT w>
- Do exercise 4.2 ncert and its examples.