SESSION 2020-21
CLASS 12 MATHS
DETERMINANTS
PART 4

By Mathematics Department

BBPS, Pitampura

Minors and Cofactors

- Watch the video
  https://youtu.be/KMKd993vG9Q

Definition 1 Minor of an element a_{ij} of a determinant is the determinant obtained by deleting its ith row and jth column in which element a_{ij} lies. Minor of an element a_{ij} is denoted by M_{ij}.

Remark Minor of an element of a determinant of order $n (n \geq 2)$ is a determinant of order $n - 1$.

Definition 2 Cofactor of an element a_{ij}, denoted by A_{ij} is defined by

$$A_{ij} = (-1)^{i+j} M_{ij}, \text{ where } M_{ij} \text{ is minor of } a_{ij}.$$

Example:
https://youtu.be/qyZR96pskJk

Note:

$$\Delta = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$= a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}$, where A_{ij} is cofactor of a_{ij}

= sum of product of elements of R_1 with their corresponding cofactors

Similarly, Δ can be calculated by other five ways of expansion that is along R_2, R_3, C_1, C_2 and C_3.

Hence Δ = sum of the product of elements of any row (or column) with their corresponding cofactors.

Note If elements of a row (or column) are multiplied with cofactors of any other row (or column), then their sum is zero. For example,

$$\Delta = a_{11} A_{21} + a_{12} A_{22} + a_{13} A_{23}$$

$$= a_{11} (-1)^{1+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{12} (-1)^{1+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} (-1)^{1+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0 \text{ (since } R_1 \text{ and } R_2 \text{ are identical)}$$

Similarly, we can try for other rows and columns.

Task

- Do exercise 4.4 ncert and its examples.

Adjoint of a Matrix

- Watch the videos:

https://youtu.be/oHzpMgKuI9Q

Example: https://youtu.be/hiuqyvR-f_4

Definition 3 The adjoint of a square matrix $A = [a_{ij}]_{n \times n}$ is defined as the transpose of the matrix $[A_{ij}]_{n \times n}$, where A_{ij} is the cofactor of the element a_{ij}. Adjoint of the matrix A is denoted by *adj* A.

Let
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then
$$adj\, A = \text{Transpose of} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Remark For a square matrix of order 2, given by

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The *adj* A can also be obtained by interchanging a_{11} and a_{22} and by changing signs of a_{12} and a_{21}, i.e.,

$$adj\, A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Change sign Interchange

Singular & Non Singular Matrices

Definition 4 A square matrix A is said to be singular if $|A| = 0$.

For example, the determinant of matrix $A = \begin{matrix} 1 & 2 \\ 4 & 8 \end{matrix}$ is zero

Hence A is a singular matrix.

Definition 5 A square matrix A is said to be non-singular if $|A| \neq 0$

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Then $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0$.

Hence A is a nonsingular matrix

Watch the video:

https://youtu.be/2OJJhfKwrRc

Properties of Adjoint

Watch the playlist to understand and learn all the important properties of determinants:

https://www.youtube.com/playlist?list=PLCx8kkz30dWk_DwE9FuHk1CdmANRqjjyT

If A, B are square matrices of order n and I_n is corresponding unit matrix, then

(i) $A\,(adj\ A) = |A| I_n = (adj\ A) A$ (Thus A (adj A) is always a scalar matrix)

(ii) $|adj\ A| = |A|^{n-1}$

(iii) $adj\ (adj\ A) = |A|^{n-2} A$

(iv) $|adj\ (adj\ A)| = |A|^{(n-1)^2}$

Inverse of a matrix

Theorem 4 A square matrix A is invertible if and only if A is nonsingular matrix.

Proof Let A be invertible matrix of order n and I be the identity matrix of order n. Then, there exists a square matrix B of order n such that AB = BA = I

Now $AB = I$. So $|AB| = |I|$ or $|A|\,|B| = 1$ (since $|I|=1, |AB|=|A||B|$)

This gives $|A| \neq 0$. Hence A is nonsingular.

Conversely, let A be nonsingular. Then $|A| \neq 0$

Now $A\,(adj\ A) = (adj\ A)\,A = |A|\,I$ (Theorem 1)

or $A\left(\frac{1}{|A|} adj\ A\right) = \left(\frac{1}{|A|} adj\ A\right) A = I$

or $AB = BA = I$, where $B = \frac{1}{|A|} adj\ A$

Thus A is invertible and $A^{-1} = \frac{1}{|A|} adj\ A$

Inverse of 2x2 matrix using adjoint: https://youtu.be/HYWeEx21WWw

Inverse of 3x3 matrix using adjoint: https://youtu.be/xfhzwNkMNg4

Verify the inverse of matrix in above 2 videos using elementary row operations (seen in previous chapter)

Task

- Do exercise 4.5 ncert and its examples.